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MTI Output Detection Statistics for Target Plus Log-Normal Clutter.

RONALD L. FANTE

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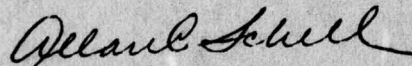
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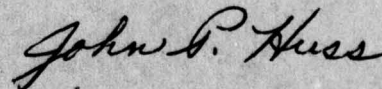
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18. SUPPLEMENTARY NOTES

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20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

We have derived the probability densities and detection statistics at the output of a quadrature-channel, two pulse canceller for the case when the input to the canceller consists of a target return immersed in log-normally distributed clutter. Analytical expressions are obtained for the probability density distribution, and these are shown to be significantly different from the log-normal or any other standard distribution. It is shown that the probability of false alarm is given, approximately, by

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$$P_f = \left(\frac{1}{1 + \beta - \ell n \beta} \right)^{1/2} \left(\frac{\beta}{\nu Z_o} \right)^{1/\nu} \exp \left\{ - \frac{1}{8\sigma^2} \left[\beta - \ell n \left(\beta - \ell n \beta \right) \right]^2 \right\}$$

where σ^2 is the variance of the input clutter distribution, $\nu = 4\sigma^2$, $\beta = \ell n (\nu Z_o)$ and Z_o is the normalized clutter residue threshold. Z_o is defined as $Z_o = r_o{}^2 I / 2\eta_o$, where r_o is the threshold of the residue voltage, η_o is the median clutter amplitude at the input to the MTI and I is the MTI improvement factor

We also show that for large improved signal-to-clutter ratios the probability of detection for a fluctuating target is

$$P_d \simeq \exp \left(- Z_o / 4\gamma \right),$$

where γ is the signal-to-clutter ratio at the output of the MTI.

Contents

Illustrations

MTI Output Detection Statistics for Target
Plus Log-Normal Clutter

1. INTRODUCTION

Generally, analyses of the detection statistics for targets immersed in clutter[1-6] do not include the effect of an MTI on the probabilities of detection and false alarm. This is certainly acceptable for targets immersed in Rayleigh clutter, because the clutter output from the MTI is still Rayleigh, except with a different variance. However, it is unacceptable for log-normal clutter. That is, if the input to the MTI is log-normally distributed the residue at the MTI output is not log-normally distributed. In this report we will therefore derive the detection statistics for the output of a two-pulse MTI canceller, when the input to the MTI consists of a target plus log-normally distributed clutter. Because of the mathematical complexity we shall limit our analyses to the two-pulse canceller, but the generalization to the N-pulse canceller is conceptually straightforward.

1. Marcum, J., and Swerling, P. (1960) Studies of target detection by pulsed radar, IRE Trans. Inform. Theory, IT-6.

2. Trunk, G., and George, S. (1970) Detection of targets in non-Gaussian sea clutter, IEEE Trans. Aerosp. Electron. Syst. AES-6:620.

3. Schleher, D. (1975) Radar Detection in Log-Normal Clutter, presented at the IEEE Int. Radar Conf.

4. Fante, R. (1977) Probability of detecting a fluctuating target immersed in both noise and clutter, IEEE, Trans. Aerosp. Electron. Syst. AES-13:711-716.

5. Trunk, G. (1971) Further results on the detection of targets in non-Gaussian sea clutter, IEEE Trans. Aerosp. Electron. Syst. AES-7:553.

6. Barton, D. (1975) Radars, Volume 5, Artech House, Dedham, MA.

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As a model for our canceller we shall consider the quadrature channel system shown in Figure 1. For purposes of review we will first derive the residue probability density function for the case when the clutter amplitude is Rayleigh. We will then consider the more difficult case of log-normally distributed clutter, and will derive single-hit detection statistics (that is, probabilities of false alarm and detection) for the cases of a constant target and a Rayleigh distributed target.
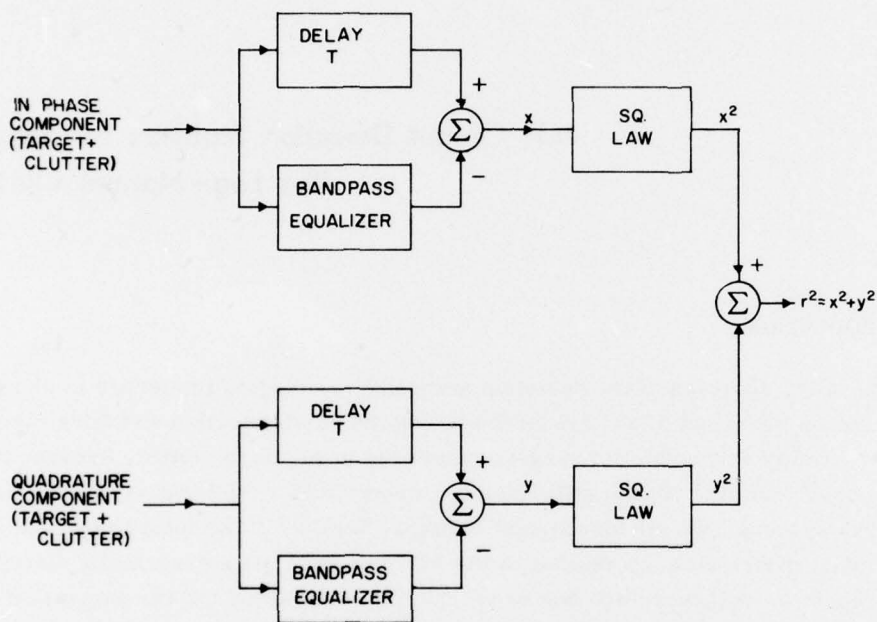
Figure 1. Quadrature Channel Canceller

2. ANALYTICAL PRELIMINARIES

The return at time t from a target immersed in clutter can be written for narrow-band signals, as

$$z(t) = s \exp\left[j(\omega_d t + \phi)\right] + C \exp\left[j(\omega_o t + \psi)\right]$$ (1)

where s and ϕ are the amplitude and phase of the voltage produced by the target, ω_d is the doppler frequency of the target, C and ψ are the amplitude and phase of the voltage produced by the clutter and ω_o is the doppler frequency of the clutter. At a later time $t + T$ the received signal can be written as

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$$z(t+T) = s' \exp\left[j\omega_d(t+T) + j\phi'\right] + C' \exp\left[j\omega_o(t+T) + j\psi'\right] \tag{2}$$

where s' and ϕ' are the target voltage amplitude and phase at time* $t+T$. If we assume that the target's amplitude and phase do not vary from pulse-to-pulse, so that $s' = s$ and $\phi' = \phi$, we then can readily see that the output of a two-pulse canceller is

$$\Delta = z(t) - z(t+T) = 2\,s\,\sin\frac{\omega_d\,T}{2}\,\exp\left[j\left(\phi + \omega_d t + \frac{\omega_d}{2}\,T - \frac{\pi}{2}\right)\right]$$
$$+ (C\,e^{j\psi} - C'\,e^{j\psi'})\,\exp(j\omega_o t) . \tag{3}$$

We now assume that $T = \pi/\omega_d$, set $\phi + \omega_d t = 2n\pi$, and also assume that there is no clutter motion, so that $\omega_o = 0$. We can then write the in-phase and quadrature components of $\Delta = r\exp(j\theta)$ as

$$r\cos\theta = 2\,s + C\cos\psi - C'\cos\psi' \tag{4}$$
$$r\sin\theta = C\sin\psi - C'\sin\psi' . \tag{5}$$

Our task now is to determine the probability densities of the residue amplitude, r, and phase θ, in terms of the probability distributions of the target and clutter. We will first consider the rather simple case when the in-phase and quadrature components of the clutter are gaussian random variables, and then proceed to the more complex case of log-normally distributed clutter.

3. RAYLEIGH CLUTTER

Let us study the case when the clutter voltage amplitude is a Rayleigh random variable. This implies that the in-phase and quadrature components of the clutter are joint-gaussian random variables. Let us rewrite Eqs. (4) and (5) as

$$x \equiv r\cos\theta = 2\,s + C_x - C_x' , \tag{6}$$

$$y \equiv r\sin\theta = C_y - C_y' . \tag{7}$$

Then if C_x, C_y, C_x', C_y' are zero-mean gaussian random variables we can write their joint probability density as

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*Also $C' \equiv C(t+T)$, $\psi' = \psi(t+T)$.

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$$p_c(C_x,\ C_y,\ C_x',\ C_y') = (2\pi\sigma_1)^{-2}\,(1-\rho_1^2)^{-1}\,\exp\left\{-\frac{1}{2\sigma_1^2(1-\rho^2)}\left[C_x^2+C_y^2\right.\right.$$

$$\left.\left.+\ C_x'^2+C_y'^2-2\rho_1 C_x C_x'-2\rho_p C_y C_y'\right]\right\}\ ,\tag{8}$$

where

$$\rho_1 \equiv \frac{\left\langle C_x C_x'\right\rangle}{\left\langle C_x^2\right\rangle} = \frac{\left\langle C_y C_y'\right\rangle}{\left\langle C_y^2\right\rangle}\ ,$$

$$\sigma_1^2 = \left\langle C_x^2\right\rangle = \left\langle C_y^2\right\rangle = \left\langle C_x'^2\right\rangle \approx \left\langle C_y'^2\right\rangle\ .$$

In writing Eq. (8) we have assumed that the in-phase and quadrature components are uncorrelated. The justification for this assumption is given in Beckmann.[7] The probability density of (x, y) keeping s fixed is then given by[8]

$$p(x,y\,|\,s) = \int_{-\infty}^{\infty} dC_x'\int_{-\infty}^{\infty} dC_y'\ p_c\left[C_x = x-2s+C_x',\ C_y = y+C_y',\ C_x',\ C_y'\right].\tag{9}$$

Upon substituting Eq. (8) into Eq. (9), and performing the integrations, we get

$$p(x,y\,|\,s) = \frac{1}{4\pi\sigma_1^2(1-\rho)}\ \exp\left\{-\frac{(x-2s)^2+y^2}{4(1-\rho_1)\sigma_1^2}\right\}\ .\tag{10}$$

We now recall that $x = r\cos\theta$, $y = r\sin\theta$. Therefore $p(r,\theta\,|\,s)$ is

$$p(r,\theta\,|\,s) = \frac{r}{4\pi\sigma_1^2(1-\rho_1)}\ \exp\left\{-\left(\frac{r^2+4s^2-4rs\cos\theta}{4\sigma_1^2(1-\rho_1)}\right)\right\}\ .\tag{11}$$

If we integrate Eq. (11) on θ from $-\pi$ to π we then get[*]

$$p(r\,|\,s) = \frac{r}{2\sigma_1^2(1-\rho_1)}\ \exp\left\{-\frac{r^2+4s^2}{4\sigma_1^2(1-\rho_1)}\right\}\ I_0\left(\frac{rs}{\sigma_1^2(1-\rho_1)}\right)\ .\tag{12}$$

7. Beckmann, P. (1967) Probability in Communication Engineering, Harcourt, Brace, and Jovanovich, New York.

8. Davenport, W., and Root, W. (1958) Random Signals and Noise, McGraw-Hill New York.

[*]Note that s is the signal level assuming the target return is solely at frequency ω_0. If, as is often done, the target signal is uniformly averaged over radial velocities, then we must replace s by $s/(2)^{1/2}$ in Eq. (12), and σ_o^2 by $\sigma_o^2/2$ in Eq. (14). The factor of 2 comes from averaging $\sin^2(\omega_d T/2)$.

Equation (12) is the probability density of the residue amplitude for a constant target amplitude. If the target amplitude fluctuates and these fluctuations obey a a probability density

$$p_t(s) = \frac{s}{\sigma_o^2} \exp\left(-\frac{s^2}{2\sigma_o^2}\right) \tag{13}$$

we find that

$$p(r) = \int_o^\infty p(r|s) p(s) ds = \frac{r \exp\left\{-\dfrac{r^2}{4[\sigma_1^2(1-\rho_1) + 2\sigma_o^2]}\right\}}{2[\sigma_1^2(1-\rho_1) + 2\sigma_o^2]} \, . \tag{14}$$

The quantity $(1-\rho_1)^{-1}$ has been defined by Nathanson[9] as the clutter improvement factor I. Consequently, for Rayleigh clutter, we see from Eqs. (12) and (14) that the effect of the two-pulse canceller is to reduce the clutter variance by the clutter improvement factor $I = [1-\rho_1(T)]^{-1}$. The results in Eqs. (12) and (14) are not unexpected, and have been presented only for completeness.

4. LOG-NORMALLY DISTRIBUTED CLUTTER

In order to consider the case when the clutter is log-normally distributed let us write $C(t) = \exp(\chi)$ and $C' = C(t+T) = \exp(\chi')$, so that Eqs. (4) and (5) become

$$r \cos \theta = 2s + e^\chi \cos \psi - e^{\chi'} \cos \psi' \, , \tag{15}$$

$$r \sin \theta = e^\chi \sin \psi - e^{\chi'} \sin \psi' \, . \tag{16}$$

We now assume that* χ, χ', ψ and ψ' are jointly distributed gaussian random variables. That is, if we define $y_1 = \chi$, $y_2 = \psi$, $y_3 = \chi'$ and $y_4 = \psi'$ their joint probability density is given by[8]

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9. Nathanson, F. (1969) Radar Design Principles, McGraw-Hill, New York.

*Note, that this differs considerably from Rayleigh clutter. In that case $C_x = \exp(\chi) \cos \psi$, $C_y = \exp(\chi)$ and ψ and so on, were gaussian random variables. Here, it is χ and ψ themselves which are gaussian.

$$p(y_1, y_2, y_3, y_4) = \frac{\exp\left\{-\frac{1}{2|\Lambda|}\sum_{n=1}^{4}\sum_{m=1}^{4}\Lambda_{nm}(y_n - \langle y_n \rangle)(y_m - \langle y_m \rangle)\right\}}{(2\pi)^2 |\Lambda|^{1/2}} \qquad (17)$$

where

$$\Lambda = \begin{bmatrix} \lambda_{11} \cdots \lambda_{14} \\ \cdot \\ \cdot \\ \cdot \\ \lambda_{41} \cdots \lambda_{44} \end{bmatrix},$$

$$\lambda_{nm} = \left\langle (y_n - \langle y_n \rangle)(y_m - \langle y_m \rangle) \right\rangle.$$

$|\Lambda|_{nm}$ is the cofactor of the element λ_{nm} in the determinant $|\Lambda|$ of the covariance matrix, and $\langle \rangle$ denotes an ensemble average. The form of the probability distribution of the residue amplitude, r, and phase θ, for the general case when χ, χ' ψ and ψ' satisfy Eq. (17) is exceedingly complex. The form of these results can be simplified considerably if we make the following assumptions:

(1) The mean phase angle of the clutter return is zero. That is,
$$\langle \psi \rangle = \langle \psi' \rangle = 0,$$

(2) The phase ψ and log-amplitude χ are uncorrelated. That is,
$$\langle (\chi - \langle \chi \rangle) \psi \rangle = \langle \psi - \langle \langle \psi \rangle \rangle \rangle \langle \psi \rangle = 0,$$

(3) The correlation function $\rho(T)$ is the same for both the log-amplitude and phase fluctuations. That is,
$$\langle (\chi - \langle \chi \rangle)(\chi' - \langle \chi' \rangle) \rangle = \sigma_\chi^2 \rho(T),$$
$$\langle \psi \psi' \rangle = \sigma_\psi^2 \rho(T).$$

In those cases where the aforementioned assumptions may be made, Eq. (17) reduces to

$$p(\chi, \chi', \psi, \psi') = p_1(\chi, \chi') p_2(\psi, \psi') \qquad (18)$$

where

$$p_1(\chi, \chi') = \frac{\exp\left\{-\frac{[(\chi - \langle \chi \rangle)^2 - 2\rho(\chi - \langle \chi \rangle)(\chi' - \langle \chi' \rangle) + (\chi' - \langle \chi' \rangle)^2]}{2\sigma_\chi^2(1-\rho^2)}\right\}}{2\pi\sigma_\chi^2(1-\rho^2)^{1/2}},$$

$$(19)$$

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$$p_2(\psi, \psi') \qquad \frac{\exp\left\{-\left[\dfrac{\psi^2 - 2\rho\,\psi\psi' + \psi'^2}{2\sigma_\psi^2(1-\rho^2)}\right]\right\}}{2\pi\sigma_\psi^2(1-\rho^2)^{1/2}},\qquad (20)$$

and the ranges of the phase angles are $-\infty \le \psi \le \infty$ and $-\infty \le \psi' \le \infty$. Ordinarily, we limit the phase to the intervals $-\pi \le \psi' \le \pi$, $-\pi \le \psi \le \pi$, rather than to the range $-\infty$ to ∞. If we do this, we have shown in Appendix A that for $\rho \simeq 1$, Eq. (20) becomes

$$p_2(\psi, \psi') \simeq \frac{\exp\left\{-\dfrac{(\psi - \psi')^2}{4(1-\rho)\sigma_\psi^2}\right\}}{4\pi^{3/2}\sigma_\psi(1-\rho)^{1/2}},\qquad (21)$$

where now $-\pi \le \psi \le \pi$, $-\pi \le \psi' \le \pi$.

Now that $p(\chi, \chi', \psi, \psi')$ is known, the next step is to determine $p(r, \theta, \chi', \psi'|s)$. In order to do this we first define $\chi' = \ell n\, v$ and set $\langle\chi\rangle = \langle\chi'\rangle = \ell n\, \eta_o$. Also we use Eqs. (15) and (16) to express χ and ψ in terms of r, θ, v and ψ' for a given signal level s. We get

$$\chi = \frac{1}{2}\ell n\, g,\qquad (22)$$

$$g = [r^2 + 4s^2 + v^2 + 2rv\cos(\theta - \psi') - 4sr\cos\theta - 4sv\cos\psi'],\qquad (23)$$

$$\psi \equiv h = \tan^{-1}\left\{\frac{r\sin\theta + v\sin\psi'}{r\cos\theta - 2s + v\cos\psi'}\right\},\qquad (24)$$

and

$$d\chi\, d\psi\, d\chi'\, d\psi' = \frac{r\, dr\, d\theta\, dv\, d\psi'}{v\, g}.\qquad (25)$$

If we use the results of Eqs. (22) to (25) in Eqs. (18) to (20) we find, assuming that the correlation coefficient $\rho(T) \simeq 1$,

$$p(r, \theta, v, \psi'|s) = r\exp\left\{-\frac{1}{4(1-\rho)\sigma_\chi^2}\left[\left(\ell n\frac{g^{1/2}}{\eta_o} - \ell n\frac{v}{\eta_o}\right)^2\right.\right.$$

$$\left.\left. + 2(1-\rho)\ell n\frac{v}{\eta_o}\ell n\frac{g^{1/2}}{\eta_o}\right] - \frac{(h-\psi')^2}{4(1-\rho)\sigma_\psi^2}\right\}\left[2(2\pi)^{5/2}(1-\rho)\sigma_\chi^2\sigma_\psi\, g\, v\right]^{-1}.$$

$$(26)$$

Finally, we obtain the probability density for the residue, r (holding s fixed)

$$p(r|s) = \frac{r}{2(2\pi)^{5/2}(1-\rho)\sigma_\chi^2 \sigma_\psi} \int_{-\pi}^{\pi} d\psi' \int_{-\pi}^{\pi} d\theta \int_0^\infty \frac{dv}{gv}$$

$$\cdot \exp\left\{ -\frac{1}{4(1-\rho)\sigma_\chi^2} \left[\left(\ln\frac{g^{1/2}}{\eta_o} - \ln\frac{v}{\eta_o} \right)^2 + 2(1-\rho)\ln\frac{v}{\eta_o}\ln\frac{g^{1/2}}{\eta_o} \right] \right.$$

$$\left. -\frac{1}{4\sigma_\psi^2(1-\rho)} (h-\psi')^2 \right\} \; . \tag{27}$$

In order to perform the integrations in Eq. (27) we first realize that because $1-\rho \simeq 0$ the quantities $[4\sigma_\chi^2(1-\rho)]^{-1}$ and $[4\sigma_\psi^2(1-\rho)]^{-1}$ in the exponential are very large. Consequently the integrand is extremely small unless $\left(\ln\frac{g^{1/2}}{\eta_o} - \ln\frac{v}{\eta_o} \right)^2$ and $(h-\psi')^2$ are quite small. This condition occurs only for $v >> r$ and $v >> s$, so that we may expand $(h-\psi')^2$ and $(\ln\frac{g^{1/2}}{\eta_o} - \ln\frac{v}{\eta_o})^2$ in a Taylor series in the small quantities r/v and s/v. If we ignore higher order terms in these small quantities we find

$$h-\psi' \simeq \frac{r}{v}\sin(\theta-\psi') + \frac{2s}{v}\sin\psi' \; , \tag{28}$$

$$\ln\frac{g^{1/2}}{\eta_o} - \ln\frac{v}{\eta_o} \simeq \frac{r}{v}\cos(\theta-\psi') - \frac{2s}{v}\cos\psi' \; , \tag{29}$$

$$g \simeq v^2 \; . \tag{30}$$

If we now use Eqs. (28) to (30) in Eq. (27), and further assume, for mathematical convenience, that

$$\sigma_\chi^2 = \sigma_\psi^2 = \sigma^2 \tag{31}$$

we obtain (again ignoring higher order terms in r/v, s/v)

$$p(r|s) \simeq \frac{r}{2(2\pi)^{5/2}(1-\rho)\sigma^3} \int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} d\psi' \int_o^\infty \frac{dv}{v^3} \exp\left\{ -\frac{1}{4\sigma^2(1-\rho)} \left[\frac{r^2}{v^2} \right. \right.$$

$$\left. \left. + 4\frac{s^2}{v^2} - \frac{4sr}{v^2}\cos\theta + 2(1-\rho)\ln^2\frac{v}{\eta_o} \right] \right\} \; . \tag{32}$$

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The ψ' integration in Eq. (32) is trivial, and the integral on θ is immediately recognized as the definition of the modified Bessel function $I_0(\ldots)$. Consequently, if we do these integrations, and then make the transformation of variables $y = \eta_o^2/v^2$ we obtain

$$p(r|s) = \frac{r}{4(2\pi)^{1/2}\sigma^3\eta_o^2(1-\rho)} \int_0^\infty dy\, I_0\left[\frac{r s y}{\sigma^2\eta_o^2(1-\rho)}\right]$$

$$\cdot \exp\left\{-\frac{1}{4\sigma^2(1-\rho)}\left[\left(\frac{r^2+4s^2}{\eta_o^2}\right)y + \frac{(1-\rho)}{2}\ln^2 y\right]\right\}. \tag{33}$$

Finally, we define new normalized residue and signal variables

$$R \equiv \frac{r}{2\eta_o\sigma(1-\rho)^{1/2}}, \tag{34}$$

$$S = \frac{s}{2\eta_o\sigma(1-\rho)^{1/2}}. \tag{35}$$

In terms of these new variables Eq. (33) assumes the simpler form

$$p(R|S) = \frac{R}{(2\pi)^{1/2}\sigma} \int_0^\infty dy\, I_0(4S\,Ry)\exp\left\{-(R^2+4S^2)y - \frac{\ln^2 y}{8\sigma^2}\right\},$$
$$0 \le R \le \infty. \tag{36}$$

Equation (36) is the normalized two-pulse canceller residue-amplitude distribution for a constant target immersed in log-normal clutter.

We can also obtain the residue probability density for a fluctuating target. For example, suppose the target fluctuations[*] have a probability density ($0 \le s < \infty$)

$$p_t(s) = \frac{s}{\sigma_o^2}\exp\left(-\frac{s^2}{2\sigma_o^2}\right). \tag{37}$$

---

[*]We emphasize that S is the signal level assuming the target return is at ω_d only. If the target were uniformly averaged over radial velocities we would replace S^2 by $S^2/2$ in Eq. (36). Likewise in Eqs. (39) and (40), we would replace σ_o^2 by $\sigma_o^2/2$.

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Then in terms of S

$$p_t(S) = \frac{4(1-\rho)\eta_o^2 \sigma^2}{\sigma_o^2} \ S \exp \left\{ - \frac{2(1-\rho)\eta_o^2 \sigma^2 S^2}{\sigma_o^2} \right\} . \tag{38}$$

Upon multiplying Eq. (36) by Eq. (38) and then integrating on S from 0 to ∞ we obtain

$$p(R) = \frac{R}{(2\pi)^{1/2}\sigma} \int_0^\infty \frac{dy \exp \left\{ \frac{R^2 y}{1+4\gamma y} - \frac{\ln^2 y}{8\sigma^2} \right\}}{1 + 4\gamma y} , \tag{39}$$

where

$$\gamma \equiv \frac{\sigma_o^2}{2(1-\rho)\eta_o^2 \sigma^2} . \tag{40}$$

Equation (39) is the probability density of the normalized two-pulse canceller-residue amplitude for a fluctuating target immersed in log-normal clutter.

For mathematical convenience we next assume that the residue voltage is passed through a square-law detector. If we define the output of the square-law detector as $Z = R^2$ and also define $W = S^2$ then, for a constant target, the square-law detector output amplitude satisfies a probability density function

$$p(Z|W) = \frac{1}{2\sigma(2\pi)^{1/2}} \int_0^\infty dy \ I_0[4y(ZW)^{1/2}] \ \exp \left\{ - (Z+4W)y - \frac{\ln^2 y}{8\sigma^2} \right\} . \tag{41}$$

For a fluctuating target, the square-law detector output Z has a probability density

$$p(Z, \gamma) = \frac{1}{2(2\pi)^{1/2}\sigma} \int_0^\infty \frac{dy \exp \left\{ \frac{Zy}{1+4\gamma y} - \frac{\ln^2 y}{8\sigma^2} \right\}}{1 + 4\gamma y} . \tag{42}$$

Before presenting numerical results, it is informative to express Z, W and γ in terms of the clutter improvement factor, I. In Appendix B we show that I is related to the variance, σ^2, and correlation coefficient ρ, of the log-amplitude fluctuation as

$$I = 2 C_A \cong \frac{1}{2\sigma^2(1-\rho)} . \tag{43}$$

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where C_A is the clutter attenuation factor, as defined by Nathanson[9]. Consequently, we can write $Z = R^2 = (r^2/2\eta_o^2)$ I, $W = (s^2/2\eta_o^2)$ I and $\gamma = (\sigma_o/\eta_o)^2$ I, where η_o is the median clutter amplitude at the input to the two-pulse canceller. Thus γ represents the improved signal to median-clutter ratio for a fluctuating target at the center doppler frequency $\omega_d = \pi/T$.

The integral in Eq. (42) has been evaluated numerically for a number of different values of γ, and the results are presented in Figure 2. We observe that the target free ($\gamma = 0$) residue probability density function has a very different behavior than that for the case of large signal-to-clutter ratios ($\gamma >> 1$). This will be clearly evident in the next section, where we evaluate $p(Z, \gamma)$ in Eq. (42) approximately. Similar results for the case of a constant target (that is, from Eq. (41)) are shown in Figure 3.

It is also important to obtain results for the probabilities of false alarm and detection. The probability of false alarm is defined as

$$P_f = \int_{Z_o}^{\infty} dZ\, p(Z, \gamma = 0)\,, \tag{44}$$

and for a fluctuating target is plotted in Figure 4 for the case when $\sigma = 0.707$.
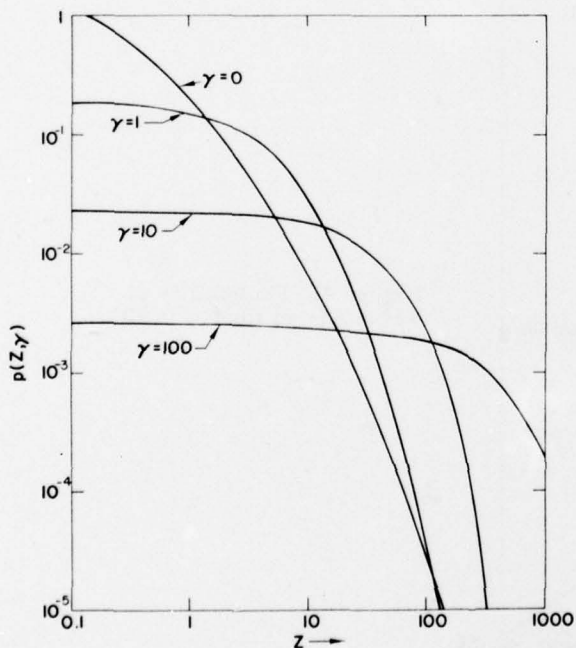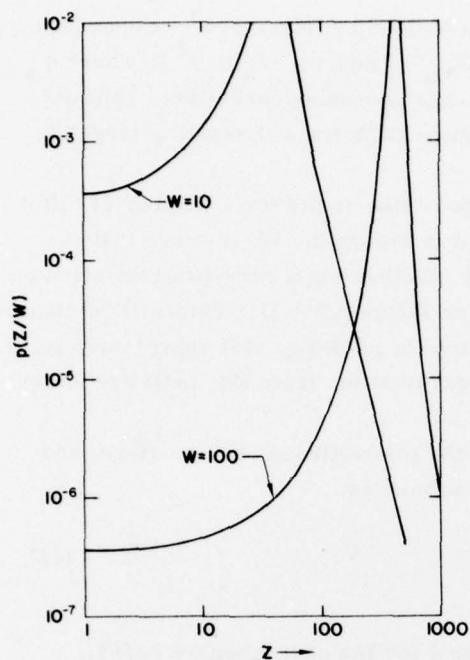

Figure 2. $p(Z, \gamma)$ for $\sigma = 0.707$
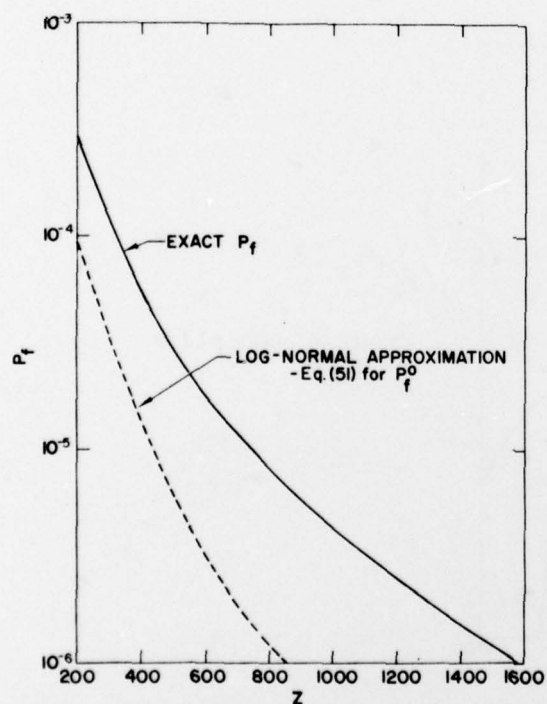
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Figure 3. p(Z/W) for σ = 0.707

Figure 4. Probability of False Alarm for σ = 0.707

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The probability of detection is defined via

$$P_d = \int_{Z_O}^{\infty} dZ \ p(Z, \ \gamma) \ , \tag{45}$$

where $p(Z, \ \gamma)$ is given by Eq. (42). Results for P_d are plotted in Figure 5, for the case when $\sigma = 0.707$.
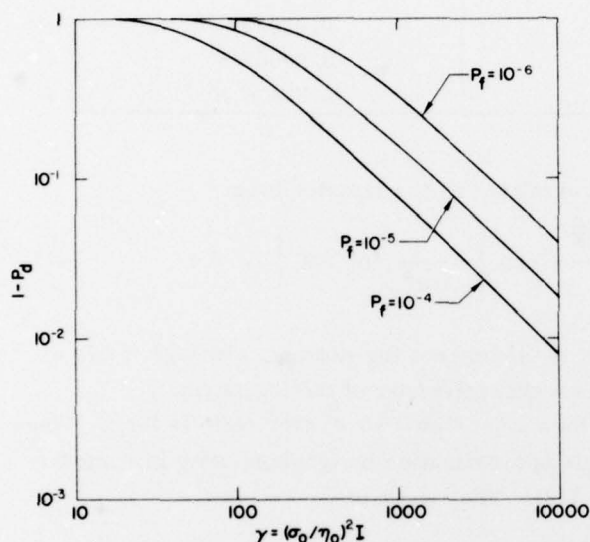
Figure 5. Probability of Detection for $\sigma = 0.707$

5. APPROXIMATE FORMULAE FOR A FLUCTUATING TARGET

In this section we will present approximate formulae for $p(Z, \ \gamma) \ P_f$ and P_d. When the target is absent ($\gamma = 0$) it is possible to evaluate Eq. (42) for large values of the normalized residue, Z, by the method of steepest descent. The result is

$$p(Z, \ \gamma = 0) = \frac{e^{-1}}{2\sigma} \left(\frac{\nu}{\nu + 1 + \ln \nu Z - \ln \xi} \right)^{1/2} \left(\frac{\xi}{\nu Z} \right)^{1/\nu} \left(\frac{\xi - \ln \xi}{\nu Z} \right)$$

$$\cdot \ \exp \left\{ - \frac{1}{8 \sigma^2} \ \ln^2 \ \left[\frac{\xi - \ln \xi}{\nu Z} \right] \right\} \ , \tag{46}$$

where $\nu = 4 \sigma^2$ and $\xi = \nu + \ln \nu Z$. A comparison of results obtained from Eq. (46) with the exact computer evaluation of Eq. (42) is given in Table 1, for the case when $\sigma = 0.707$. Observe that $Z > 1$, Eq. (46) is a very good approximation to Eq. (42).

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Table 1. Comparison of the Approximate Result in Eq. (46) With the Exact Evaluation of Eq. (42)

Z	$p(z, \gamma = 0)$ – from Eq. (42)	$p(z, \gamma = 0)$ – from Eq. (46)
0.1	1.2512	0.9131
0.3	0.6456	0.6667
1	0.2134	0.2197
3	0.0517	0.0528
10	0.006669	0.006760
30	0.000634	0.000639
100	0.0000275	0.0000276
300	9.294×10^{-7}	9.294×10^{-7}

For $Z \to \infty$, Eq. (46) can be approximated by the simpler form

$$p(Z, \gamma = 0) \to \frac{e^{-1} (\ln \nu Z)^{\frac{1}{\nu} + \frac{1}{2}}}{(\nu Z)^{1 + 1/\nu}} \exp \left\{ - \frac{1}{8\sigma^2} \ln^2 \nu Z \right\} . \tag{47}$$

Thus, even for $Z \to \infty$ the residue pdf is clearly not log-normal, although it has a relatively slow dropoff rate which is a characteristic of the log-normal.

Next we consider the pdf when the output signal-to-clutter ratio is large. When $\gamma \gg 1$ we can again obtain an analytic approximation by ignoring unity in comparison with $4\gamma y$ in the integral in Eq. (42). The result is

$$p(Z, \gamma \gg 1) \simeq \frac{1}{4\gamma} \exp \left(- \frac{Z}{4\gamma} \right) . \tag{48}$$

Results obtained using Eq. (48) are compared with the exact evaluation of the integral in Eq. (42) in Table 2. Note that the agreement is quite good provided $Z \ll 4\gamma^2$.

A comparison of Eqs. (46) and (48) clearly shows that the nature of the residue probability density function is quite different for large values of target-signal to clutter ratio than for the case when the target is absent; in the former case the pdf is exponential whereas in the latter case, it is a variant of the log-normal.

We have also obtained simplified expressions for the probabilities of false alarm and detection. For large thresholds $(Z_o \gg 1)$ we find

$$P_f \simeq \left(\frac{1}{1 + \beta - \ln \beta} \right)^{1/2} \left(\frac{\beta}{\nu Z_o} \right)^{1/\nu} \exp \left\{ - \frac{1}{8\sigma^2} \left[\beta - \ln (\beta - \ln \beta) \right]^2 \right\} \tag{49}$$

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where $\beta = \ell n \, (\nu \, Z_o)$ and $\nu = 4\sigma^2$. Comparisons of the results obtained using Eq. (49) with exact computer calculations of P_f are given in Table 3. We observe that the agreement is excellent.

Table 2. Comparison of the Approximate Result in Eq. (48)
With the Exact Evaluation of Eq. (42)

Z	$\gamma = 10$		$\gamma = 100$	
	$p(Z, \gamma)$ exact	$p(Z, \gamma)$ from Eq. (48)	$p(Z, \gamma)$ exact	$p(Z, \gamma)$ from Eq. (48)
0.1	0.0237	0.0250	0.00248	0.00250
1	0.0231	0.0243	0.00248	0.00249
10	0.0186	0.0195	0.00242	0.00244
100	0.00222	0.00205	0.00194	0.00195
300	0.0000245	0.0000138	0.00118	0.00118
1000			0.000207	0.000205

Table 3. Comparison of the Exact Values of P_f With the
Approximate Results Obtained Using Eq. 49) for
$\sigma = 0.707$

Z_o	P_f (exact)	P_f (approximate – from Eq. (49)
314	1.0001×10^{-4}	1.0070×10^{-4}
740	1.0072×10^{-5}	1.0120×10^{-5}
1591	1.0013×10^{-6}	1.0044×10^{-6}

When Z_o is sufficiently large and $\nu = 4\sigma^2$ is of order unity, we can approximate Eq. (49) further as

$$P_f \rightarrow (\nu \, Z_o)^{-1/\nu} \, (\ell n \, Z_o)^{\frac{1}{\nu} - \frac{1}{2}} \exp \left\{ - \frac{1}{8\sigma^2} \ell n^2 \, Z_o \right\} \quad . \tag{50}$$

It is interesting to speculate what results we would have obtained if we had simply assumed that the residue clutter distribution were log-normal, with the same variance as the input clutter distribution, but with the median level squared, η_o^2, divided by the clutter attenuation factor $C_A = I/2$. Then upon using Eq. (30) in Reference 4, with η_o^2 replaced by $2 \eta_o^2/I$, we would find

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$$P_f^o = \sigma \left(\frac{2}{\pi}\right)^{1/2} \frac{\exp\left\{-\frac{1}{8\sigma^2}\ln^2 Z_o\right\}}{\ln(Z_o)} .$$ (51)

A comparison of the result in Eq. (51) with the exact results for P_f indicates that this is a poor approximation, and leads to threshold levels which are incorrect by a factor of 2 or more.

Finally, for large values of the equivalent signal-to-clutter ratio, γ, we can derive a suitable approximation for P_d. This is

$$P_d \simeq \exp\left\{-\frac{Z_o}{4\gamma}\right\} ,$$ (52)

where Z_o is the threshold required to give a specified value of P_f. A comparison of results obtained using Eq. (52) with exact computer calculations of P_d is given in Table 4. We see that Eq. (52) is quite accurate whenever γ is sufficiently large that $P_d > 0.05$. Because this is usually the regime of interest, it is evident that Eq. (52) is a useful approximation.

Although we have plotted numerical results only for $\sigma = 0.707$, it is clear that Eq. (49) and Eq. (52) can be used to obtain results for other values of σ. Some typical results are shown in Figures 6 to 9. Observe that the detection statistics depend strongly on σ.

Table 4. Comparison of Exact Computer Calculations of P_d With the Approximation in Eq. (52) for $\sigma = 0.707$

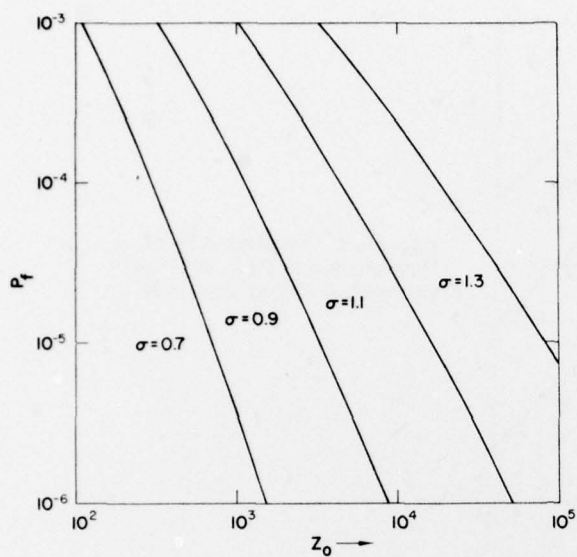|	$P_f = 10^{-4}$		$P_f = 10^{-5}$		$P_f = 10^{-6}$	
	P_d (exact)	P_d (Eq. 52)	P_d (exact)	P_d (Eq. 52)	P_d (exact)	P_d (Eq. 52)
10	0.00103	0.00039	0.0000203	9.24×10^{-9}		
30	0.0773	0.0730	0.00248	0.00210	6.52×10^{-6}	1.74×10^{-6}
100	0.458	0.456	0.1589	0.1572	0.0192	0.0187
300	0.7698	0.7698	0.540	0.540	0.266	0.266

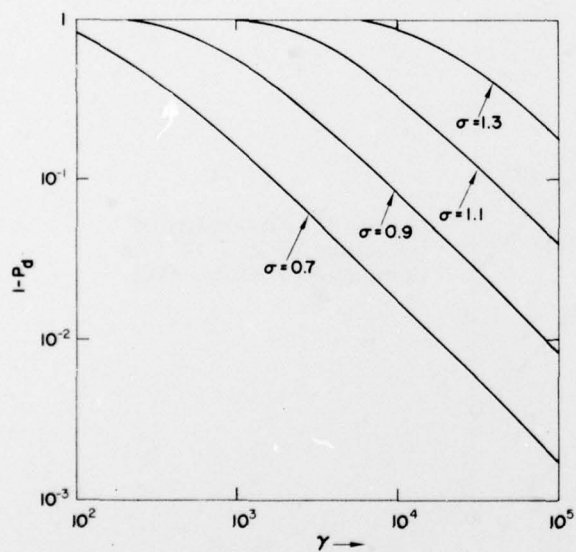Figure 6. Probability of False Alarm as Computed From Eq. (49)

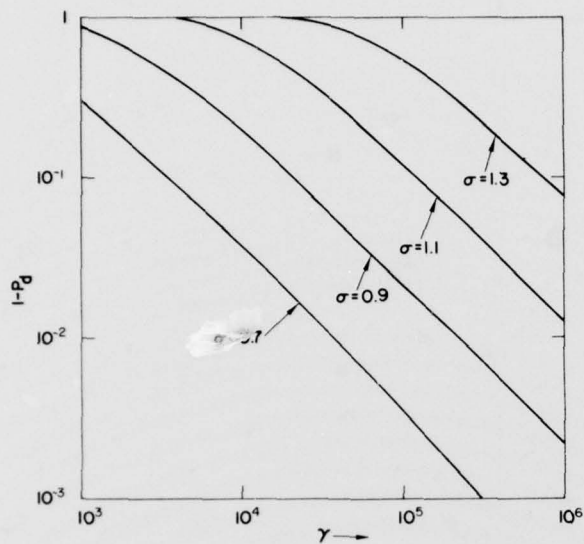Figure 7. Probability of Detection for $P_f = 10^{-5}$ as Computed From Eq. (52)

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Figure 8. Probability of Detection for $P_f = 10^{-6}$ as Computed From Eq. (52)
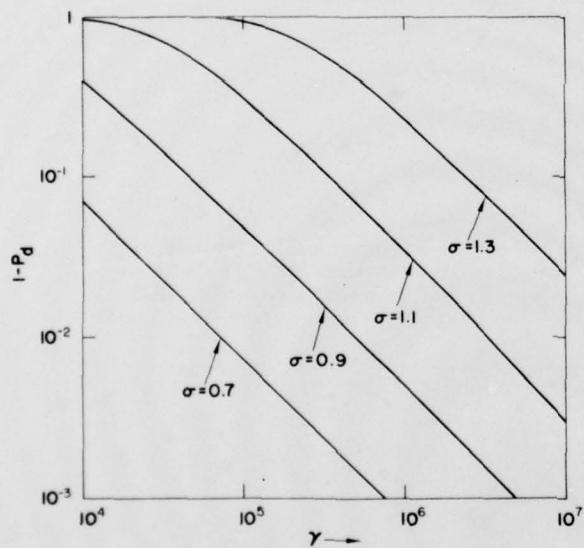

Figure 9. Probability of Detection for $P_f = 10^{-7}$ as Computed From Eq. (52)

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6. DISCUSSIONS

It is rather interesting to compare the detection statistics for log-normally distributed clutter with those for Rayleigh clutter. Suppose we assume that the median clutter level is the same for both the Rayleigh and log-normal clutter returns, and we also use the same definition of γ for the Rayleigh distribution as for the log-normal case. That is, we again define $\gamma = (\sigma_o^2/\eta_o^2) I$, where I is the clutter improvement factor. If we assume that the residue voltage in Eq. (14) is passed through a square law detector, define $R_2 = r^2 I/2\eta_o^2$, and then integrate the result to obtain P_d and P_f, we find that for a fluctuating target immersed in Rayleigh distributed clutter

$$P_d = \exp\left\{ \frac{\ln P_f}{1 + \dfrac{2\gamma \eta_o^2}{\sigma_1^2}} \right\} \qquad . \tag{53}$$

For Rayleigh-distributed clutter σ_1, and η_o are related via $\eta_o^2 = 2.7725\, \sigma_1^2$. Consequently Eq. (53) becomes

$$P_d = \exp\left\{ \frac{\ln P_f}{1 + 5.545\,\gamma} \right\} \qquad . \tag{54}$$

The result in Eq. (54) is plotted in Figure 10 for a probability of false alarm, P_f of 10^{-6}. Also plotted on the same figure is the probability of detection for the case when the clutter is log-normally distributed, with the same median value, η_o, as for the Rayleigh clutter, and variance $\sigma^2 = 0.5$. We note that, because the signal-free residue pdf and consequently the threshold, is very different there is a considerable difference in the probability of detection in log-normal as opposed to Rayleigh clutter.

Our discussion here has been limited strictly to the case of a single hit (that is, a single output from a two-pulse canceller). If the target and clutter outputs remain nearly perfectly correlated from pulse-to-pulse then the M-hit probabilities of false alarm and detection are the same as those for one hit.

We have also limited our discussion to the two-pulse canceller. The extension of our analysis to an N-pulse canceller is straightforward but extremely laborious, and will not be pursued here.
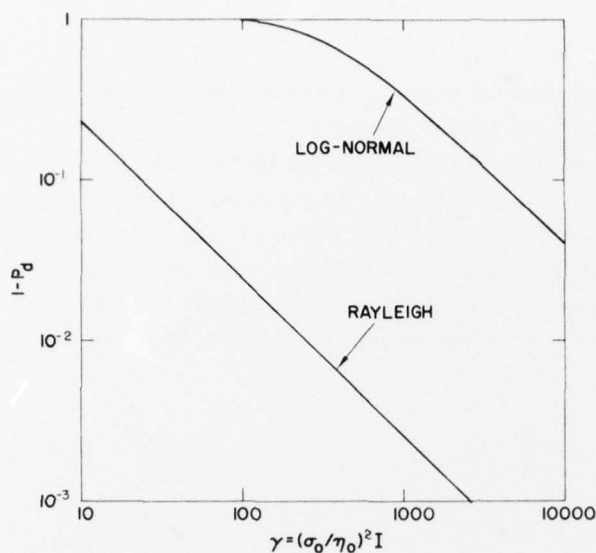
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Figure 10. Probability of
Detection for $P_f = 10^{-6}$
($\sigma = 0.707$)

7. CONCLUSIONS

We have derived the output probability densities and detection statistics for a
two-pulse canceller when the input to the canceller consists of a target immersed
in log-normally distributed clutter. We have shown that, even with the target
absent, the output residue is not log-normally distributed but has a probability den-
sity which can be approximated by Eq. (46) for large values of $Z = r^2 I/2\eta_o^2$. This
distribution, however, does have a relatively slow dropoff with the consequence that
a relatively high normalized threshold, Z_o, is required in order to achieve a given
probability of false alarm. This threshold is much higher for log-normal clutter
than it is for Rayleigh clutter.

Finally, we again emphasize that our definition of signal power level always
assumes that the return is strictly at the center frequency $\omega_d = \pi/T$. For an average
over radial velocities we would replace s^2 by $s^2/2$ in Eq. (12), Eq. (36), and so on.
Similarly in Eqs. (14), (39), (40), and so on, we would replace σ_o^2 by $\sigma_o^2/2$ to
include the effect of the velocity average.

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We consider the probability density given by Eq. (20) which is defined for ψ and ψ' ranging from $-\infty$ to ∞, and realize that if we limit the range of the phase from $-\pi$ to π then the probability of the phase lying in $d\psi\, d\psi'$ about $(\psi,\, \psi')$, where $\psi,\, \psi'$ are now limited to the range $-\pi$ to π, is equal to the probabilities of $(\psi,\, \psi')$ in Eq. (20) lying in $d\psi\, d\psi'$ about $(\psi + 2\,n\,\pi,\, \psi' + 2\,m\,\pi)$ for all possible integers n and m. That is

$$p(\psi,\, \psi') = \frac{1}{2\pi\sigma_\psi^{\,2}(1-\rho^2)} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma_\psi^{\,2}(1-\rho^2)}\left[(\psi + 2\,n\,\pi - \psi' - 2\,m\,\pi)^2\right.\right.$$
$$\left.\left. + 2(1-\rho)(\psi + 2\,n\,\pi)\,(\psi' + 2\,m\,\pi)\right]\right\}, \tag{A1}$$

where ψ and ψ' now lie between $-\pi$ and π. Now if the correlation from pulse to pulse is high, as is necessary for good MTI operation, then $\rho(T) \simeq 1$, and consequently $2(1-\rho)\sigma_\psi^{\,2} << 1$. This means that in Eq. (A1) the term $\exp\left\{-[\psi - \psi' = 2\,\pi\,(n-m)]^2/2\sigma_\psi^{\,2}(1-\rho^2)\right\}$ is negligible except when $n = m$. Therefore Eq. (A1) becomes

$$p(\psi,\, \psi') \simeq \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left\{-\frac{(\psi - \psi')^2}{2\sigma_\psi^{\,2}(1-\rho^2)}\right\} \sum_{n=-\infty}^{\infty} \exp\left\{-\frac{(\psi + 2\,n\,\pi)(\psi' + 2\,n\,\pi)}{(1+\rho)\sigma_\psi^{\,2}}\right\}.$$

$$\tag{A2}$$

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The sum in Eq. (A2) can be performed, approximately, and for $\rho \simeq 1$ we then find from Eq. (A2) that

$$p(\psi, \psi') \simeq \frac{1}{4(\pi)^{3/2}(1-\rho)^{1/2}\sigma_\psi} \ \exp \ \left\{ - \frac{(\psi - \psi')^2}{4(1-\rho)\sigma_\psi^2} \right\} \ , \tag{A3}$$

where now $-\pi \leq \psi \leq \pi$ and $-\pi \leq \psi' \leq \pi$, instead of their original range $-\infty \leq \psi \leq \infty$ and $-\infty \leq \psi' \leq \infty$.

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Appendix B

Clutter Improvement Factor for Log-Normal Clutter

The ensemble averaged residue power for a two-pulse canceller, when clutter-only is present, can be written as

$$\langle |r|^2 \rangle = \langle | C\, e^{j\psi} - C'\, e^{j\psi'} |^2 \rangle . \tag{B1}$$

Upon recalling that $C = \exp(\chi)$, $C' = \exp(\chi')$ we can rewrite Eq. (B1) as

$$\langle |r|^2 \rangle = \langle e^{2\chi} + e^{2\chi'} - 2\, \mathrm{Re}\, e^{\chi + \chi' + j\,(\psi - \psi')} \rangle . \tag{B2}$$

Because χ and ψ are gaussian random variables we can use the well-known result that, if ϕ is gaussian, then

$$\langle \exp \phi \rangle = \exp\left\{ \langle \phi \rangle + \frac{1}{2}\, \langle [\phi - \langle \phi \rangle]^2 \rangle \right\} .$$

If we normalize the clutter power so that* $\langle \exp 2\chi \rangle = 1$, and use the aforementioned result we find that Eq. (B2) becomes

$$\langle |r|^2 \rangle = 2\, \left\{ 1 - \mathrm{Re}\, \exp\left[\langle \chi + \chi' \rangle + \frac{1}{2} \left\langle \left[\chi - \langle \chi \rangle + \chi' - \langle \chi' \rangle + j(\psi - \psi') \right]^2 \right\rangle \right] \right\} . \tag{B3}$$

---

*This implies that $\langle \chi \rangle = -\sigma_\chi^2$, where $\sigma_\chi^2 \equiv \langle (\chi - \langle \chi \rangle)^2 \rangle$.

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If we now use our previous assumptions that $\langle \chi' \rangle = \langle \chi \rangle$, $\langle \psi' \rangle = \langle \psi \rangle = 0$, χ and ψ are stationary, and the fact that $\langle \chi \rangle = -\sigma_\chi^2$ we find that Eq. (B3) becomes

$$\langle |r|^2 \rangle = 2 \left\{ 1 - \exp \left[-\sigma_\chi^2 + \langle (\chi - \langle \chi \rangle)(\chi' - \langle \chi' \rangle) \rangle - \langle \psi^2 \rangle + \langle \psi \psi' \rangle \right] \right\} .$$

(B4)

Upon recalling that $\langle (\chi - \langle \chi \rangle)(\chi' - \langle \chi' \rangle) \rangle = \sigma_\chi^2 \, \rho(T)$, $\langle \psi^2 \rangle = \sigma_\psi^2$, $\langle \psi \psi' \rangle = \sigma_\psi^2 \, \rho(T)$, and finally using our previous assumption in Eq. (31) that $\sigma_\chi^2 = \sigma_\psi^2 = \sigma^2$, we may rewrite Eq. (B4) as

$$\langle |r|^2 \rangle = 2 \left\{ 1 - \exp \left[-2\sigma^2 (1 - \rho) \right] \right\} .$$

(B5)

If we remember that the average gain in a two-pulse canceller is 2 we find that the clutter improvement factor I is

$$I = \frac{1}{1 - \exp[-2\sigma^2(1-\rho)]} .$$

(B6)

Finally, because $\rho \simeq 1$ we have that $2\sigma^2(1-\rho) << 1$. Therefore we may expand $\exp[-2\sigma^2(1-\rho)]$ in a Taylor series to obtain

$$I \simeq \frac{1}{2\sigma^2(1-\rho)} .$$

(B7)